HD–TVP–97–06

Noise induced transport at zero temperature

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February 1, 2008

Abstract

We consider a particle in the over-damped regime at zero temperature under the influence of a sawtooth potential and of a noisy force, which is correlated in time. A current occurs, even if the mean of the noisy force vanishes. We calculate the stationary probability distribution and the stationary current. We discuss, how these items depend on the characteristic parameters of the underlying stochastic process. A formal expansion of the current around the white-noise limit not always gives the correct asymptotic behaviour. We improve the expansion for some simple but representative cases.

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1 Introduction

In the last two years there has been a considerable interest in the problem of noise induced transport. The motivation to study such models has been initiated by Magnasco [1]. He showed that there are two necessary ingredients for noise induced transport: A stochastic force that is correlated in time and an environment without inversion symmetry. In the one-dimensional case one can have e.g. a periodic potential without inversion symmetry. Such a potential is often called a ratchet-like potential.

Magnasco was mainly interested in the adiabatic limit, i.e. in the case where the fluctuations of the stochastic force are slow. A few month later, the problem was investigated more carefully by Doering et al. [3]. They studied the motion of a one-dimensional particle in a saw-tooth potential and in the case where the force is described by various stochastic processes. For the symmetric dichotomous Markov process and at vanishing temperature they obtained an exact solution for small correlation times. For the Ornstein–Uhlenbeck process and for a class of processes called kangaroo processes, they calculated the current to first order in the correlation time. They also presented some results from Monte-Carlo simulations of such a system, again at zero temperature. Later, Mielke [4] developed a method that allows to calculate the current for a large class of processes including the ones discussed by Doering et al. [3], again in the case of a saw-tooth potential or more generally for a piecewise linear potential. He recovered the results from [3] and found several other cases where a current reversal occurs. One of these cases was a process that consists of an even sum of dichotomous processes. For such a process his results differ from the perturbative result to first order in the correlation time [3]. The sign of the current was different, even for very small values of the correlation time. One of the motivations of the present work is this discrepancy. We will show that the current and the stationary distribution of the coordinate of the moving particle are non-analytic functions of the correlation time. To do this we generalize the method developed in [4] so that it is applicable to zero temperature as well. This allows us to obtain exact numerical results for the current and the stationary distribution, as well as a correct asymptotic expansion for small correlation times. This is also of general interest, since there are many approximative methods to solve Langevin or Fokker–Planck equations in this or other cases that agree with the usual perturbative expansion for small correlation times (for a review we refer to [5]). Furthermore we show that some of the properties of the solution for small correlation times are relevant for larger correlation times as well. Even if the solution is analytic for small correlation times, it may become non-analytic for larger correlation times. This always happens if the support of the stationary distribution of the stochastic force is finite.

The paper is organized as follows: In the next section we define the model and the stochastic processes we will be able to deal with. In the subsequent section we show how the method in [4] can be generalized so that one can treat the zero temperature case as well. Since a large part of the method is similar to the finite temperature case, we refer to [4] for details. In section 4 we present some of the numerical results including a detailed discussion of the current reversal for various noise processes. In section 5 we show how the asymptotic expansion for small correlation times can be obtained and we give some explicit results for simple noise processes. We also include a comparison between the asymptotic expansion and the exact numerical results to show the range of validity of the asymptotic expansion. Finally we give some conclusions.

2 Definition of the model

We consider the one-dimensional motion $x(t)$ of a particle in a dissipative environment. The particle moves in a one-dimensional periodic potential $V(x)$ with period L, and it is subject to a stochastic force $z(t)$ that is correlated in time. In the over-damped regime, the motion of the particle is described by a

Langevin equation of the form.

$$\frac{dx}{dt} = f(x) + \sqrt{2T}\xi(t) + z(t). \tag{2.1}$$

We use units where the friction constant is unity. The first term on the right hand side is the force $f(x) = -\frac{dV}{dx}$ due to the potential $V(x)$. The second term describes thermal fluctuations, $\xi(t)$ is a white noise with zero mean and $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. In the later part of the paper we will discuss the zero temperature case, $T = 0$. The additional noisy force $z(t)$ has zero mean, $\langle z(t)\rangle = 0$. We assume that it is described by a Markov process with an infinitesimal generator M_z. The probability density $p(z,t)$ of this Markov process satisfies the Fokker-Planck equation

$$\frac{\partial p(z,t)}{\partial t} = M_z p(z,t). \tag{2.2}$$

We discuss a class of Markov processes for which the generator M_z is described by its eigenvalues and by certain properties of its right eigenfunctions, namely

$$M_z\phi_n(z) = -\lambda_s\phi_n(z), \quad z\phi_0(z) = \gamma_{0,1}\phi_1(z),$$

$$z\phi_n(z) = \gamma_{n,n+1}\phi_{n+1}(z) + \gamma_{n,n-1}\phi_{n-1}(z) \quad n = 1,2,\ldots \tag{2.3}$$

The eigenvalues $-\lambda_n$ obey

$$\lambda_n \leq \lambda_m \quad \text{if} \quad n < m, \quad \lambda_0 = 0 \tag{2.4}$$

$\phi_0(z)$ is the stationary distribution of z. Due to the recursion relations the eigenfunctions $\phi_n(z)$ can be written as $\phi_n(z) = g_n(z)\phi_0(z)$ where $g_n(z)$ are orthogonal polynomials with respect to the weight function $\phi_0(z)$,

$$\int dz\, g_n(z) g_m(z)\phi_0(z) = \delta_{n,m}. \tag{2.5}$$

This class of Markov processes is very general. It contains many processes that occur in typical situations such as the Ornstein–Uhlenbeck process, the dichotomous process, sums of dichotomous processes, and kangaroo processes. The correlation time τ of the process is related to the smallest non-vanishing eigenvalue λ_1 of the noise process via $\tau = \lambda_1^{-1}$.

The joint probability density $\rho(x,z,t)$ for the two stochastic variables $x(t)$ and $z(t)$ obeys a Fokker-Planck equation of the form

$$\frac{\partial\rho(x,z,t)}{\partial t} = -\frac{\partial}{\partial x}\left(f(x)+z\right)\rho(x,z,t) + M_z\rho(x,z,t) \tag{2.6}$$

We discuss only stationary properties of this equation, therefore the left hand side is put to zero and (2.6) becomes an equation for the stationary probability density $\rho(x,z)$. Due to the periodicity of $V(x)$ we assume $\rho(x,z)$ to be periodic in x with period L. To solve the stationary Fokker-Planck equation it is useful to expand $\rho(x,z)$ in terms of the right eigenfunctions of M_z.

$$\rho(x,z) = p_0(x)\phi_0(z) + \sum_{n=1}^{\infty}(-1)^n\phi_n(z)p_n'(x). \tag{2.7}$$

This yields the recursion relations

$$J = f(x)p_0(x) - \gamma_{0,1}p_1'(x) \tag{2.8}$$

$$\gamma_{0,1}p_0(x) = \lambda_1 p_1(x) + f(x)p_1'(x) - \gamma_{1,2}p_2'(x) \tag{2.9}$$

$$\gamma_{n-1,n}p'_{n-1}(x) = \lambda_n p_n(x) + f(x)p'_n(x) - \gamma_{n,n+1}p'_{n+1}(x), \quad n > 1. \tag{2.10}$$

for the functions $p_n(x)$. As additional conditions we have the normalization of $p_0(x)$ and the periodicity of $p_n(x)$,

$$\int_0^L p_0(x)dx = 1, \quad p_n(x) = p_n(x+L). \tag{2.11}$$

$p_0(x)$ is the stationary distribution of $x(t)$. It is one of the quantities we are interested in. A second and more important quantity is the integration constant J in (2.8), it is the stationary current. If the potential $V(x)$ has inversion symmetry, the current vanishes. In the general situation, where $V(x)$ has no inversion symmetry, one obtains generically a non-vanishing current.

To solve the recursion relations (2.8–2.10) for a general Markov process M_z approximations are necessary. One has to truncate the recursion at some large value N, i.e. one has to put $\gamma_{N,N+1} = 0$. For some processes, e.g. for a sum of dichotomous processes, one has $\gamma_{N,N+1} = 0$ for some finite value of N and the approximation is not necessary. In principle it is then possible to solve the recursion relations numerically. Here, we restrict ourselves to a class of piecewise linear potentials. This is a standard assumption [3], such a potential is often called a ratchet-like potential. If the potential is piecewise linear, the force $f(x)$ is piecewise constant. The differential equations (2.8–2.10) are equations with constant coefficients that can be solved analytically. The remaining algebraic problem is to solve some continuity conditions for the functions $p_n(x)$ at the points where the force jumps from one constant value to another. In that way it is possible to express the stationary current as a ratio of two determinants. It can thus be calculated easily numerically for various parameters of the system or even analytically for small values of N. The general procedure has been described for finite temperatures in detail in [4]. One can obtain very accurate results in the whole parameter regime and for various noise processes. Thereby one observes that several analytical approximation schemes do not work in different regimes of the parameter space. For instance, the standard expansion for small correlation times τ breaks down. The physical reason is that for sufficiently small values of z the particle cannot escape one of the minima of the potential. At zero temperature, this is only possible if $|z|$ is larger than the largest value of $|f(x)|$. Since the process $z(t)$ has a finite correlation time, z remains for a while in the region where $|z| < |f(x)|$ and the particle cannot move apart. For a general potential $V(x)$ this yields divergencies in the stationary distribution $p_0(x)$, in the case of a piecewise linear potential, the stationary distribution contains contributions of the type $w_i\delta(x - x_i)$, where x_i are the positions of the minima of $V(x)$ and w_i are some weights. Such an effect occurs always if there is a finite probability to have a stochastic force $z(t)$, that fulfils the condition $f(x_i - 0_+) \leq z \leq f(x_i + 0_+)$. It occurs especially in the case where $\gamma_{N,N+1} = 0$ for some finite N. Mathematically it turns out that the stationary distribution $p_0(x)$ and the current J are nonanalytic functions of τ in some regions of the parameter space. This is the reason why the τ–expansion fails for some noise processes.

In the next section we describe what modifications to the method in [4] are necessary so that $T = 0$ can be treated as well. The main goal is to obtain correct zero temperature results for the stationary behaviour of the model and to derive a correct asymptotic expansion for small τ.

3 The method at zero temperature

In the rest of the paper we restrict ourselves to the simplest nontrivial case for the force $f(x)$; we assume that $f(x)$ takes to different values. To be precise, we let

$$\begin{aligned} f(x) &= f_1 \qquad \text{if} \qquad 0 \leq x < L_1, \\ f(x) &= f_2 \qquad \text{if} \qquad L_1 \leq x < L. \end{aligned} \tag{3.1}$$

We let $L_2 = L - L_1$. Due to the periodicity of $V(x)$ we have $f_1 L_1 + f_2 L_2 = 0$. We assume that $V(x)$ has a minimum at $x = 0$, which means $f_1 < 0,\ f_2 > 0$. Due to the discussion above, $p_0(x)$ has the form

$$p_0(x) = \tilde{p}_0(x) + W_0\,\delta(x). \tag{3.2}$$

$\tilde{p}_0(x)$ contains no further δ–contributions. In the same way we obtain for the functions $p'_n(x)$

$$p'_n(x) = \tilde{p}'_n(x) + W'_n\,\delta(x). \tag{3.3}$$

The coefficients W_0 and W'_n can be related to each other using the differential equations 2.8-2.10. One obtains

$$W'_{2k-1} = 0 \tag{3.4}$$
$$W'_{2k} = (-1)^k \frac{\gamma_{2k-2,2k-1}}{\gamma_{2k-1,2k}} \cdots \frac{\gamma_{0,1}}{\gamma_{1,2}} W_0 \tag{3.5}$$

For $\tilde{p}_0(x)$ and $\tilde{p}'_n(x)$ we make the ansatz

$$\tilde{p}_{0,i}(x) = \sum_r c_{r,i} a^{(r)}_{0,i} \alpha^{(r)}_i e^{(\alpha^{(r)}_i x)} + b_{0,i} \tag{3.6}$$
$$\tilde{p}_{1,i}(x) = \sum_r c_{r,i} a^{(r)}_{1,i} e^{(\alpha^{(r)}_i x)} + b_{1,i} \tag{3.7}$$
$$\tilde{p}_{n,i}(x) = \sum_r c_{r,i} a^{(r)}_{n,i} e^{(\alpha^{(r)}_i x)} + b_{n,i} \tag{3.8}$$

The index i takes the two values $i = 1, 2$, according to the two regions where $f(x) = f_i$. Inserting this ansatz in the differential equations, we obtain a generalized eigenvalue problem to determine the coefficients $\alpha^{(r)}_i$, $a^{(r)}_{n,i}$, and $b^{(r)}_{n,i}$. This is the usual procedure for a system of coupled linear differential equations with constant coefficients and has been described for the present case in detail in [4]. We obtain using a vector notation

$$\vec{\mathbf{b}}_i = \begin{pmatrix} b_{0,i} \\ b_{1,i} \\ b_{2,i} \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{J}{f_i} \\ \frac{J\gamma_{0,1}}{\lambda_1 f_i} \\ 0 \\ \vdots \end{pmatrix} \quad . \tag{3.9}$$

The generalized eigenvalue problem for $\alpha^{(r)}_i$ and $a^{(r)}_{n,i}$ is

$$\mathbf{A}_i \vec{\mathbf{a}}_i = \alpha_i \mathbf{B}_i \vec{\mathbf{a}}_i \quad . \tag{3.10}$$

where

$$\mathbf{A}_i = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & \lambda_1 & 0 & \cdots \\ 0 & 0 & \lambda_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \quad , \quad \mathbf{B}_i = \begin{pmatrix} -f_i & \gamma_{0,1} & 0 & \cdots \\ \gamma_{0,1} & -f_i & \gamma_{1,2} & \cdots \\ 0 & \gamma_{1,2} & -f_i & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \quad . \tag{3.11}$$

$\alpha^{(0)}_i$ is zero, the other eigenvalues can be determined either be explicitely solving the eigenvalue problem or by using certain continued fractions as described in [4]. To finally solve the problem, one has to determine the coefficients $c_{r,i}$, the weight W_0 of the δ–contribution in $p_0(x)$, and the current J. These are $2N + 2$ unknown variables. The continuity of $p_n(x)$ at $x = L_1$ and certain jump conditions of $p_n(x)$ at $x = 0$ involving the coefficients W'_n for $n \geq 1$ yield $2N$ equations, in addition we have the normalization

of $p_0(x)$. The last condition can be obtained from the continuity of the current density at a fixed value of z. For finite N we can write $\rho(x,z)$ in the form

$$\rho(x,z) = \sum_{k=0}^{N} P^{(k)}(x)\,\delta(z - z_k) \quad . \tag{3.12}$$

for a process $z(t)$ that takes $N+1$ different values z_k. $P^{(k)}(x)$ can be related to $p_0(x)$ and $p_n'(x)$, for instance

$$\sum_{k=0}^{N} P^{(k)}(x) = p_0(x). \tag{3.13}$$

For each $P^{(k)}(x)$ we have a corresponding current

$$J^k(x) := P^k(x)\, f^k(x). \tag{3.14}$$

We introduced $f^k(x) := f_i + z_k$ if x lies in the region where $f(x) = f_i$. The current $J^k(x)$ has to be continuous at $x = L_1$. If $|z_k| < \min_i(|f_i|)$ and $\mathrm{sign}(z_k) = -\mathrm{sign}(f_i)$ this yields $J^k(L_1) = 0$. With this equation we have found the last condition we need to determine the $2N+2$ unknown variables. In this way the calculation of the current J and the stationary distribution $p_0(x)$ has been reduced to the purely algebraic problem of solving $2N+2$ linear equations for $2N+2$ variables. As in [4], the current J can be expressed in closed form using determinants.

4 Results

The results presented in this section have been obtained by solving the algebraic problem mentioned above numerically. We present results for two classes of processes, namely for a sum of dichotomous processes and for some kangaroo processes.

4.1 Sums of dichotomous processes

A dichotomous process $z(t)$ is a process where $z(t)$ takes two values, $\pm z_0$. Summing up N such processes, we obtain a Markov process with a stationary distribution of the form

$$\phi_0(z) = \frac{1}{2^N}\left(\sum_{i=0}^{N}\binom{N}{i}\delta\left(z - (N-2i)\,z_0\right)\right). \tag{4.1}$$

The parameters that characterize the process are

$$\lambda_n = -n/\tau, \quad n = 0,\ldots,N, \tag{4.2}$$

$$\gamma_{n,n+1} = \sqrt{(n+1)(N-n)}z_0, \quad n = 0,\ldots,N. \tag{4.3}$$

In order to have a value for $\langle z^2\rangle = \gamma_{0,1}^2$ that is independent of N, we choose $z_0 = \gamma/\sqrt{N}$ where $\gamma = \sqrt{D/\tau}$. D is the noise strength of the process. In the limit $N \to \infty$, the sum of N dichotomous processes yields the Ornstein–Uhlenbeck process [6].

The eigenvalue problem (3.10) can be solved explicitely for a sum of dichotomous processes. This has been shown in [4]. The eigenvalues $\alpha_i^{(k)}$ are

$$\alpha_i^{(1)} = \frac{N\lambda f_i}{N\gamma^2 - f_i^2} \tag{4.4}$$

$$\alpha_i^{(2k),(2k+1)} = \frac{N\lambda f_i}{2}\frac{N \pm (N-2k)\sqrt{1+\frac{4k(N-k)\gamma^2}{Nf_i^2}}}{(N-2k)^2\gamma^2 - Nf_i^2}\ ,\ \text{for } 2k+1 \leq N \tag{4.5}$$

and in addition

$$\alpha_i^{(N)} = -\frac{N\lambda}{2f_i} \tag{4.6}$$

for even N. The coefficients J, W_0, and $c_{r,i}$ can be determined numerically as described above. Let us first discuss the results for small correlation times τ. Since $\gamma^2 = D/\tau$, $\lambda = 1/\tau$, the eigenvalues $\alpha_i^{(k)}$ have singularities $\propto \tau^{-1/2}$ and $\propto \tau^{-1}$. The second type of singularity is present only if N is even. Furthermore, for small τ and N odd, $|z(t)|$ is always larger than $|f(x)|$ and we expect that $W_0 = 0$, whereas for even N, $z(t)$ may be zero and therefore we expect $W_0 > 0$. The behaviour should be very different depending on whether N is even or odd. First we will show some results for the current. In Fig. 1a the current is plotted as a function of τ for N odd. The current is always positive. The corresponding result for even N is shown in Fig. 1b. The current is negative for small τ and changes the sign when τ becomes larger. This has already been observed for very low temperatures in [4]. We will come back to this point in the next section, where we discuss an expansion for small τ. Let us mention that a negative current for even N and small τ occurs only if the noise strength D is sufficiently large. For larger values of τ, the current is shown for some processes in Fig. 2. The current has cusp-like maxima that occur near such values of τ, for which one of the discrete values of $|z(t)|$ becomes smaller than one of the discrete values of $|f(x)|$. The exact position of these values of τ is indicated in the figure. It is interesting to see that the maxima don't lie exactly at this value of τ but slightly below. With increasing number of maxima, i.e. for higher values of τ this effect increases as well. Up to now we have no physical interpretation for it.

Whenever τ becomes larger than one of these values of τ, the weight W_0 of the δ–peak in the stationary distribution $p_0(x)$ jumps discontinuously to a higher value. This is shown in Fig. 3 for the same parameter as used in Fig. 2. The weight W_0 shows the expected behaviour and is unity for large τ. The coefficients $c_{r,i}$ can be calculated as well, they enter in the expression for the stationary distribution $p_0(x)$. Typical results are shown for various processes in Fig. 4. τ is very small, so that a δ–contribution (which is not shown) occurs only for even N. One observes that for both cases, even or odd values of N, the stationary distribution $p_0(x)$ has discontinuities. We will come back to that point later. The minimum at $x = 0.8$ is located at the maximum of the potential. It is strong for $N = 2$ and becomes weaker when N is even and increases. The opposite behaviour is found for odd N, here the minimum is only weak for $N = 1$ and becomes more and more pronounced for larger N. Similarly, the maximum at $x = 0$ increases with increasing N if N is odd.

4.2 Kangaroo processes

Kangaroo processes are processes with $\lambda_n = -1/\tau$ for all $n > 0$. They can be completely characterized by the stationary distribution $p_{st}(z)$. In this section we discuss results for kangaroo processes with a stationary distribution given in (4.1). According to the notations there, we call these processes $K(N)$. This allows a direct comparison with the results for the sums of dichotomous processes. The only difference between the kangaroo processes and the sums of dichotomous processes is that for the latter case, only jumps of a magnitude $\pm z_0$ in $z(t)$ occur, whereas a kangaroo process has no restriction of the jumps of $z(t)$. Since the stationary distributions are the same, the parameters $\gamma_{n,n+1}$ are the same. But unfortunately, one cannot derive an analytic expression for the eigenvalues $\alpha_i^{(r)}$ for $N > 4$. They have to be determined numerically, which is easily done.

We show some of the numerical results for the current as a function of the correlation time τ in Fig. 5a for N odd and Fig. 5b for N even. The inset in Fig. 5a shows that the current is negative for very small

τ. This is in contrast to the sums of dichotomous processes. Furthermore, the region of negative current becomes larger, when D becomes smaller. For even N the current reversal is observed as for the sum of dichotomous processes, but the region of negative current is larger. Furthermore the current is always negative for small τ and N even, whereas for the sum of dichotomous processes a negative current occurs only if the noise strength is large. Thus, the behaviour of the current as a function of τ and D differs significantly from that one of the sums of dichotomous processes, although the stationary distribution is the same in both cases. Let us mention that the mechanism that produces a negative current for small τ and N odd is not clear. Doering et al. [3] proposed that a negative current occurs due to a parameter called flatness, which is a property of the stationary distribution. Their argument cannot be applied in the present case, since the stationary distributions and therefore the flatness is the same for the kangaroo processes and the sums of dichotomous processes, whereas the sign of the current for small τ is different.

5 An expansion for small τ

The usual formal expansion for small τ has already been discussed by Doering et al [3]. It can be obtained using a standard perturbational treatment [7] to solve the stationary Fokker–Planck equation. The unperturbed part is M_z, which is of order τ^{-1}. The perturbation contains a term proportional to z, it is thus $\propto \tau^{-1/2}$. But since it contains only off-diagonal matrix elements in the basis in which M_z is diagonal (there are no parameters $\gamma_{n,n}$), one obtains finally an expansion in powers of τ for the stationary distribution and for the current J. Since the current vanishes in the white noise limit, the first term in the expansion for J vanishes and $J \propto \tau$. An alternative way to obtain the same expansion is to use a formal operator continued fraction that can be obtained from the recursion relations (2.8-2.10). This has been shown in the appendix of [4], where a general expression for the first order has been given and higher orders can be obtained straight forward. In our special case of a saw–tooth potential one obtains

$$J = \tau J_1 + \tau^2 J_2 + \dots \tag{5.1}$$

with

$$J_1 = \left(1 - \frac{\lambda_1}{\lambda_2}\frac{\gamma_{1,2}^2}{\gamma_{0,1}^2}\right) \frac{f_1^2 - f_2^2}{D^3\left(\frac{1}{f_1} - \frac{1}{f_2}\right)^2 \left(e^{\frac{1}{D}} - 1\right)\left(e^{\frac{-1}{D}} - 1\right)} \tag{5.2}$$

$$\begin{aligned} J_2 \quad = \quad & \left(1 - \frac{\lambda_1}{\lambda_2}\frac{\gamma_{1,2}^2}{\gamma_{0,1}^2}\right) \frac{f_1^2 - f_2^2}{N_{inh}^{(1)} D^2 \left(\frac{1}{f_1} - \frac{1}{f_2}\right)\left(e^{\frac{1}{D}} - 1\right)} \\ & + \left(2\frac{\gamma_{1,2}^2}{\gamma_{0,1}^2}\frac{\lambda_1}{\lambda_2} + \frac{\lambda_1^2}{\lambda_2^2}\frac{\gamma_{1,2}^2}{\gamma_{0,1}^2} - \frac{\lambda_1^3}{\lambda_2^2\lambda_3}\frac{\gamma_{2,3}^2\gamma_{1,2}^2}{\gamma_{0,1}^4} - \frac{\lambda_1^2}{\lambda_2^2}\frac{\gamma_{1,2}^4}{\gamma_{0,1}^4} - 1\right) \\ & * \frac{f_1^4 - f_2^4}{D^4\left(\frac{1}{f_1} - \frac{1}{f_2}\right)^2\left(e^{\frac{1}{D}} - 1\right)\left(e^{\frac{-1}{D}} - 1\right)}. \end{aligned} \tag{5.3}$$

This result generalizes the result given by Doering et al. [3]. It can be compared with the exact results of the sums of dichotomous processes and for the kangaroo processes shown in the last section. One observes that the formal expansion is correct for a sum of an odd number of dichotomous processes, if the correlation time τ is sufficiently small. For a sum of an even number of dichotomous processes the result is not even qualitatively correct, the current has the wrong sign. Similar results hold for the

kangaroo processes. If N is odd, the expansion is correct for small τ, whereas it is wrong for even N. The reason is that the stationary distribution $p_0(x)$ contains a contribution $W_0\delta(x)$ for even N and arbitrary small τ, but not for odd N. This leads to non-analytic contributions to $p_0(x)$ and J as functions of τ if N is even, which are not taken into account in the perturbative τ–expansion. This expansion is well defined only if $p_0(x)$ is a differentiable function of x, which is true for odd N, but not for even N.

A second possibility to obtain an asymptotic τ–expansion for J and $p_0(x)$ is to start from the set of linear equations that determine the variables J, W_0, and $c_{r,i}$. The coefficients in this set of linear equations contain singularities of the type

$$e^{+\frac{c}{\sqrt{\tau}}} \quad \text{(type 1)}$$

$$e^{-\frac{c}{\sqrt{\tau}}} \quad \text{(type 2)}$$

$$e^{-\frac{c}{\tau}} \quad \text{(type 3)}$$

where c is some constant. These singularities occur due to the behaviour of the eigenvalues $\alpha_i^{(k)}$ for small τ, see e.g. (4.4-4.6). For small τ it is possible to divide those equation, which contain singularities of type 1 by $\exp(+c/\sqrt{\tau})$. We can then neglect all the terms that vanish faster than any power of $\sqrt{\tau}$. For small N it is possible to solve the remaining set of linear equations analytically. For a single dichotomous process this yields

$$J^{(1)}(\tau) = J_1^{(1)}\tau + J_2^{(1)}\tau^2 + O(\tau^3) \tag{5.4}$$

where

$$J_1^{(1)} = \frac{f_1^2 - f_2^2}{D^3\left(\frac{1}{f_1}-\frac{1}{f_2}\right)^2\left(e^{\frac{1}{D}}-1\right)\left(e^{\frac{-1}{D}}-1\right)} \tag{5.5}$$

$$\begin{aligned} J_2^{(1)} = {} & \frac{f_2^4 + f_1^4}{D^6\left(\frac{1}{f_1}-\frac{1}{f_2}\right)^2\left(e^{\frac{1}{D}}-1\right)^2\left(e^{-\frac{1}{D}}-1\right)^2} \\ & + \frac{1}{2}\frac{\left(f_2^4+f_1^4\right)\left(1-e^{\frac{2}{D}}\right)}{D^5\left(\frac{1}{f_1}-\frac{1}{f_2}\right)^2\left(e^{\frac{1}{D}}-1\right)^4} \\ & - \frac{f_1^4 - f_2^4}{D^4\left(\frac{1}{f_1}-\frac{1}{f_2}\right)^2\left(e^{\frac{1}{D}}-1\right)\left(e^{-\frac{1}{D}}-1\right)}. \end{aligned} \tag{5.6}$$

As expected, this result agrees with the formal τ–expansion described above, which was correct for a single dichotomous process. For a sum of two dichotomous processes we obtain

$$J^{(2)} = \frac{1}{2}J_1^{(1)}(1-D)\tau + O(z^3). \tag{5.7}$$

This expression differs from the formal τ–expansion, but it agrees well with the exact numerical results for small τ. The current is negative for small τ and $D > 1$. For $N = 3$ one would again expect a result that

agrees with the formal τ–expansion, but this is not true. We obtain

$$J^{(3)} = \frac{1}{3}J_1^{(1)}\tau + J_{3/2}^{(3)}\tau^{\frac{3}{2}} + O(z^4) \tag{5.8}$$

$$J_{3/2}^{(3)} = -\frac{1}{2\sqrt{6}}D^{-\frac{5}{2}}\frac{f_1^2 f_2^2 (f_1 - f_2)}{\left(e^{\frac{1}{D}} - 1\right)\left(e^{\frac{-1}{D}} - 1\right)}. \tag{5.9}$$

The additional term $\propto \tau^{\frac{3}{2}}$ was not present in the formal τ–expansion. It is responsible for the very small region of validity of the formal τ–expansion for $N = 3$. Similar terms occur for any odd N. The reason is that the stationary distribution $p_0(x)$ is not a differentiable function of x. It has discontinuities at the extrema of the potential, see Fig. 4. Therefore the perturbative τ–expansion is not well defined.

Finally one can look at the kangaroo process $K(2)$, for which the formal τ–expansion was incorrect too. We obtain

$$J^{K(2)} = J_1^{K(2)}\tau + O(z^3) \tag{5.10}$$

$$J_1^{K(2)} = -DJ_1^{(1)}, \tag{5.11}$$

which is in agreement with the exact numerical results. A comparison between the numerical results and the asymptotic expansions is shown in Fig. 6.

6 Conclusions

We discussed in detail the behaviour of the static properties of one-dimensional models for noise induced transport at zero temperature. For our discussion we used a sawtooth potential, but the method and the results can be generalized easily to any piecewise linear potential. If the support of the stationary distribution of the stochastic force $z(t)$ is finite, one observes non-analytic contributions in the induced current and in the stationary distribution of the coordinate of the particle. In a piecewise linear potential, the stationary distribution of the coordinate $p_0(x)$ contains contributions of the form $w_i\delta(x - x_i)$. Such contributions occur for sufficiently large values of τ, for some processes even for all $\tau > 0$. If the $\phi_0(z)$ and the force $f(x) = -dV/dx$ are continuous, one can not expect a δ–contribution to $p(x)$. But even then one expects singularities in $p(x)$. Let $f_{\min}$ and $f_{\max}$ be the minimum and the maximum of $f(x)$, and let us assume without loss of generality that $|f_{\min}| < |f_{\max}|$. Let us now take a stochastic force $z(t)$ that takes only a finite, discrete set of values z_i. If $0 < z_i < -f_{\min}$, the particle moves to the right until $z_i = f(x)$. At these special values of x singularities in $p(x)$ may occur, depending on the behaviour of $f(x)$ near such a point. A similar situation occurs if $0 > z_i > -f_{\max}$. In the more general case where the stochastic force can take all values within an interval $[-z_0, z_0]$, similar singularities may occur, again depending on the form of $f(x)$. Therefore the results we presented are relevant for a large class of situations. Let us mention that one should not expect such singularities in the case of a Gaussian noise process, since in that case the stochastic force may be arbitrarily large.

A consequence of the non-analytic behaviour of $p(x)$ is that the usual perturbative τ-expansion breaks down. In the n-th order of this expansion a derivative of order $(n+2)$ of $p(x)$ occurs. If this derivative doesn't exist, it is clear that the τ-expansion is not defined. In contrary, the asymptotic τ-expansion we derived for some special cases is always well defined.

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Figure captions

Fig. 1. The current J as a function of the correlation time τ for a sum of N dichotomous processes. a) N odd and $D = 10\Delta V$ and $L_1/L_2 = 4$.
b) The same as in a) for N even.

Fig. 2. The stationary current for the sum of $N = 1,2,3$ dichotomous processes as a function τ on a larger range of τ. The tags τ_{Nk} indicate the correlation times, for which is $kz_0 = f_2$. The curves for $N = 1,2,3$ have maxima near $\tau_{1k}, \tau_{2k}, \tau_{3k}$. The parameters are the same as in Fig. 1.

Fig. 3. The weight of the δ distribution as a function of τ for the sum of N dichotomous processes for the same parameters as in Fig. 1.

Fig. 4. The stationary probability distribution of the position coordinate in a normed period and with $L_1/L_2 = 4$ for the sum of N dichotomous processes. $N = 1$ (solid line), $N = 2$ (dashed line) $N = 3$ (short dashed line) and $N = 4$ (long dashed line). The δ–contribution is not shown.

Fig. 5. a) The current J as a function of the correlation time τ for the kangaroo processes $K(N)$, N, odd with the same parameters as in Fig. 1. The inset shows the same plot on a finer scale.
b) The same as in a) for N even.

Fig. 6. Comparison of the linear approximation (solid lines) of the current as a function of the correlation time with the exact results (dashed lines) for the sum of N dichotomous processes and the kangaroo process $K(2)$.

Fig. 1.a)

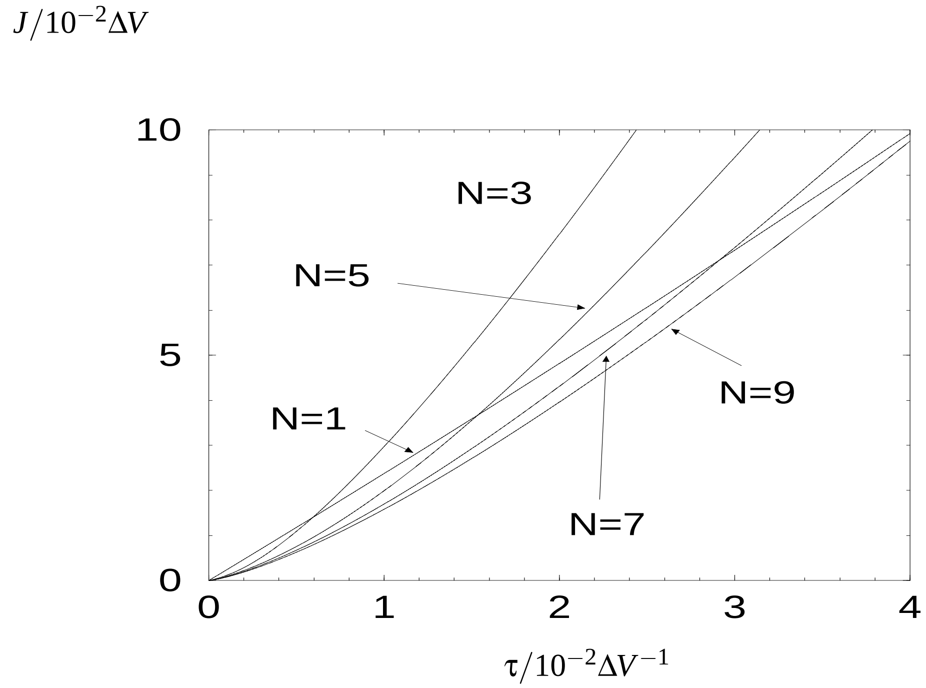

$J/10^{-2}\Delta V$
10
5
0
0
1
2
3
4
N=3
N=5
N=1
N=9
N=7
$\tau/10^{-2}\Delta V^{-1}$

Fig. 1.b)

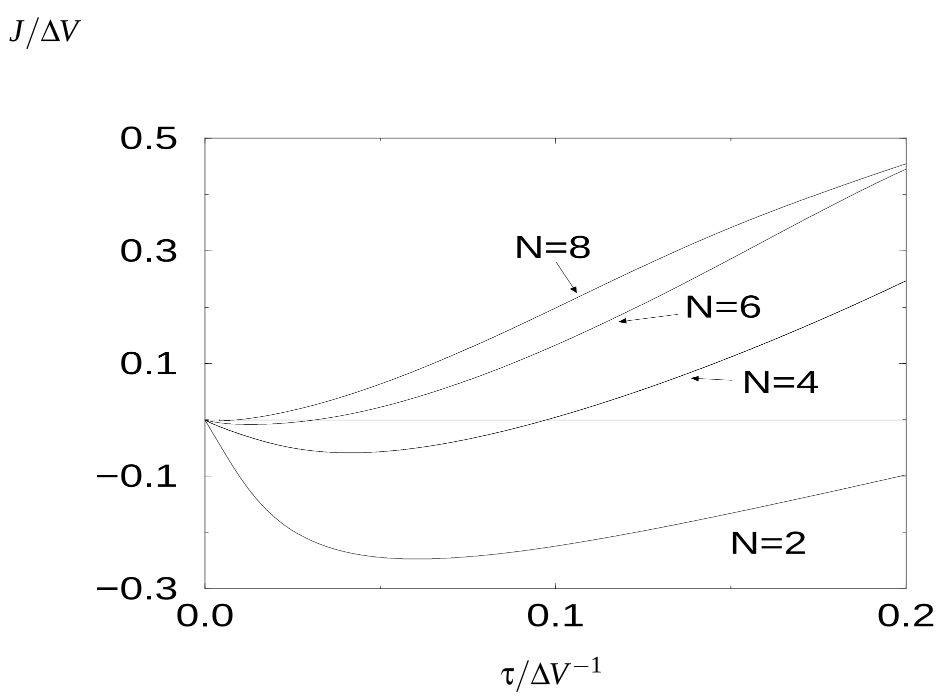
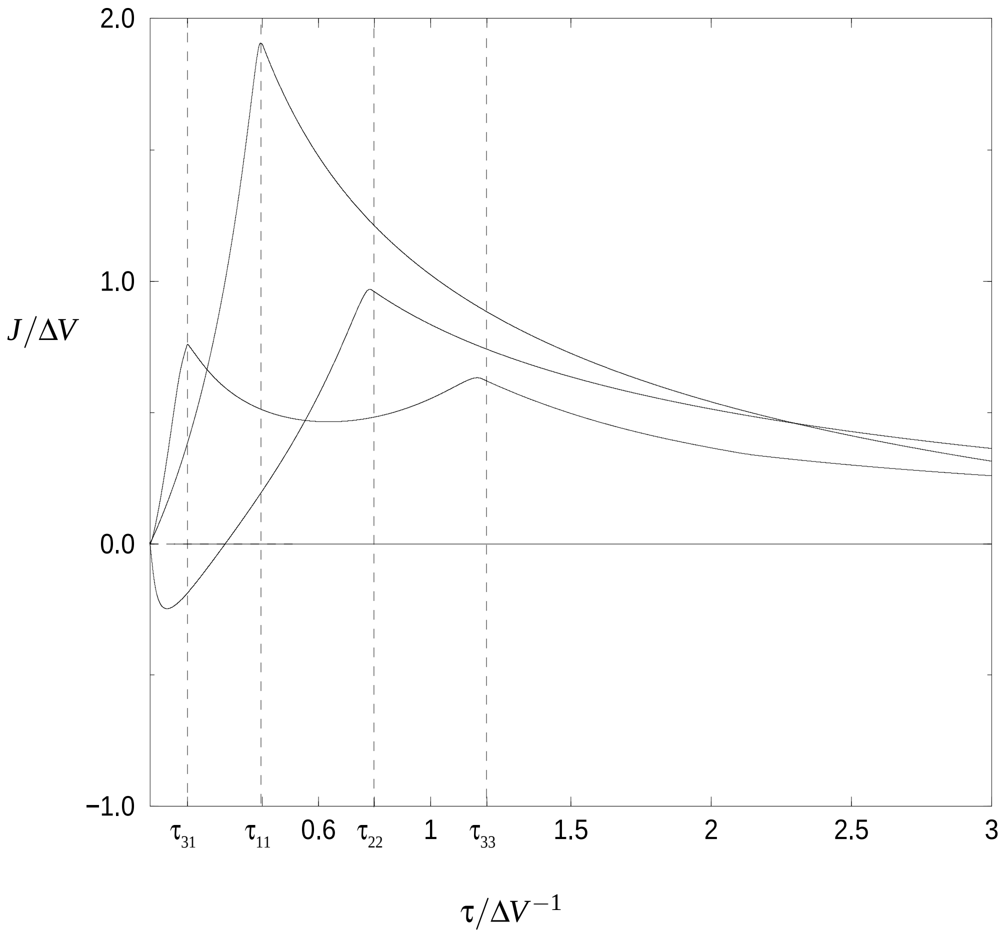
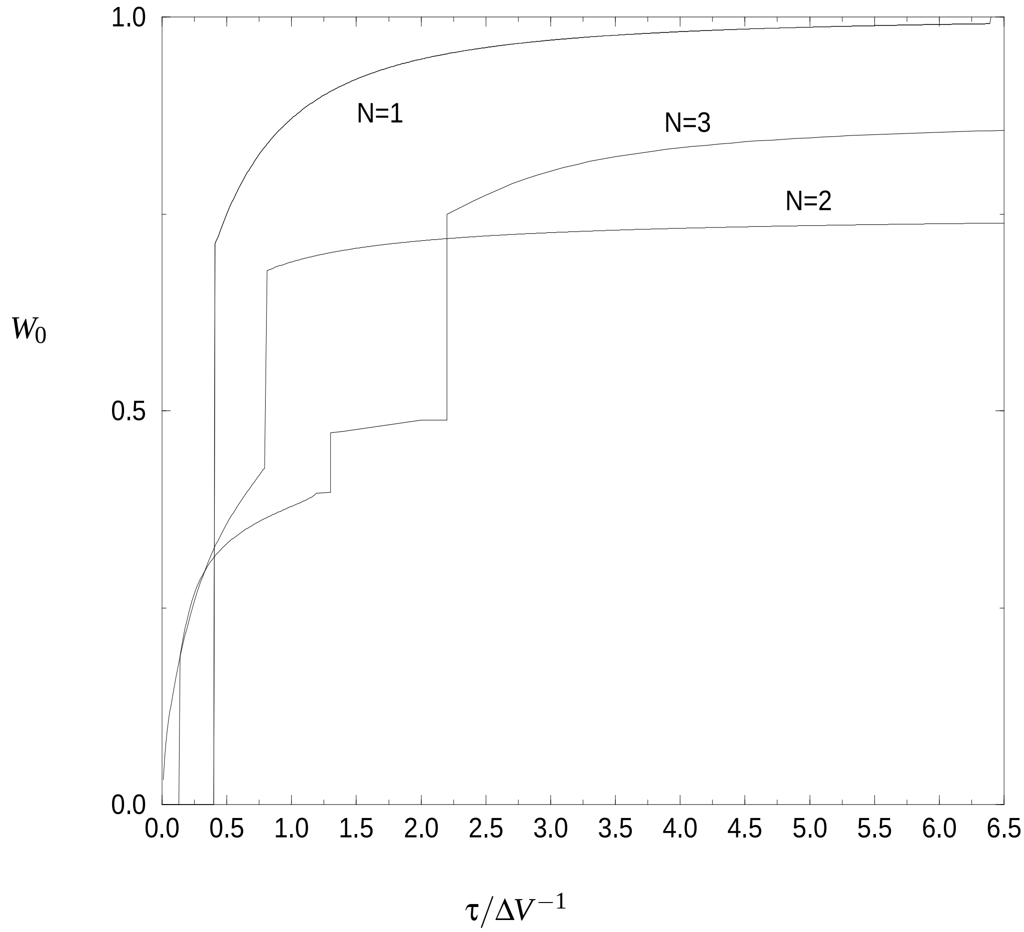

$J/\Delta V$
0.5
0.3
0.1
−0.1
−0.3
0.0
0.1
0.2
N=8
N=6
N=4
N=2
$\tau/\Delta V^{-1}$

Fig. 2



2.0
1.0
0.0
−1.0
$J/\Delta V$
τ_{31}
τ_{11}
0.6
τ_{22}
1
τ_{33}
1.5
2
2.5
3
$\tau/\Delta V^{-1}$

Fig. 3



1.0
0.5
0.0
N=1
N=3
N=2
W_0
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0 5.5 6.0 6.5
$\tau/\Delta V^{-1}$

Fig. 4

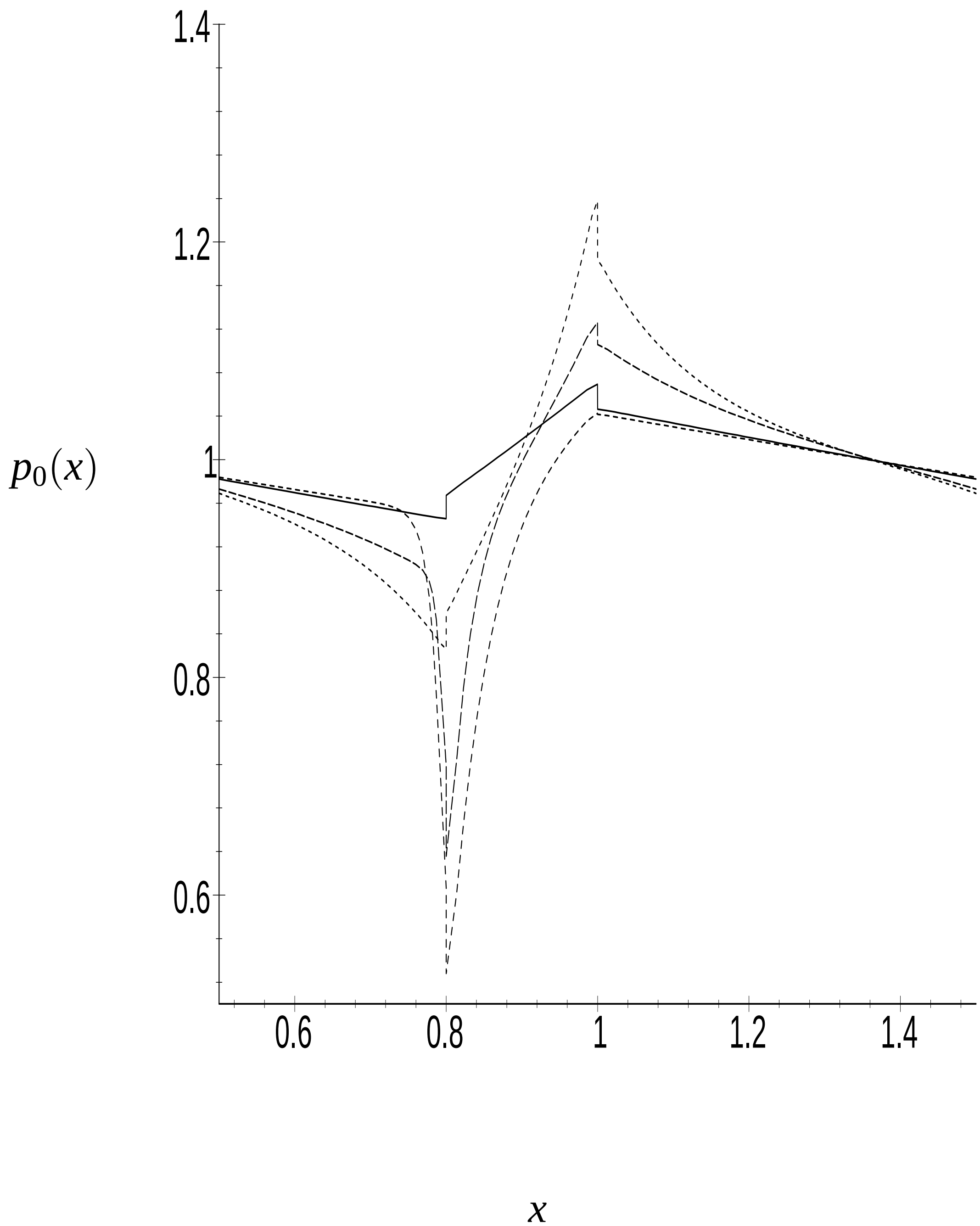
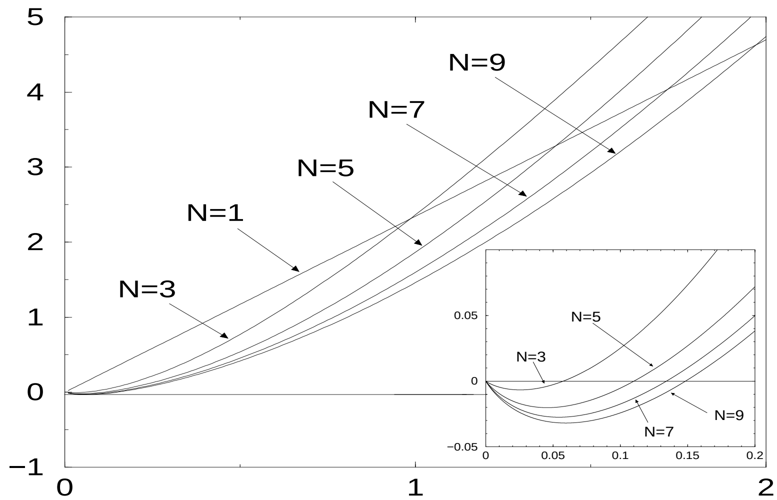


Fig. 5a)

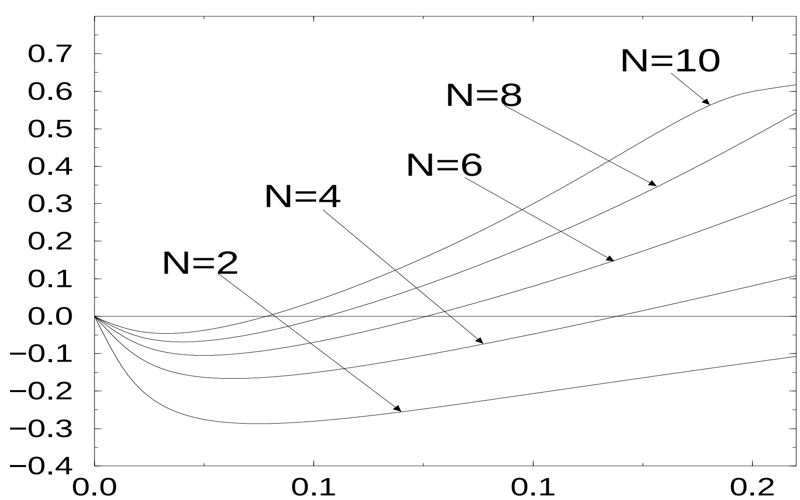
$J/10^{-3}\Delta V$



$\tau/10^{-3}\Delta V^{-1}$

Fig. 5b)

$J/\Delta V$



$\tau/\Delta V^{-1}$

Fig. 6